Markov chain of binary sequences generated by A/D conversion using β -encoder

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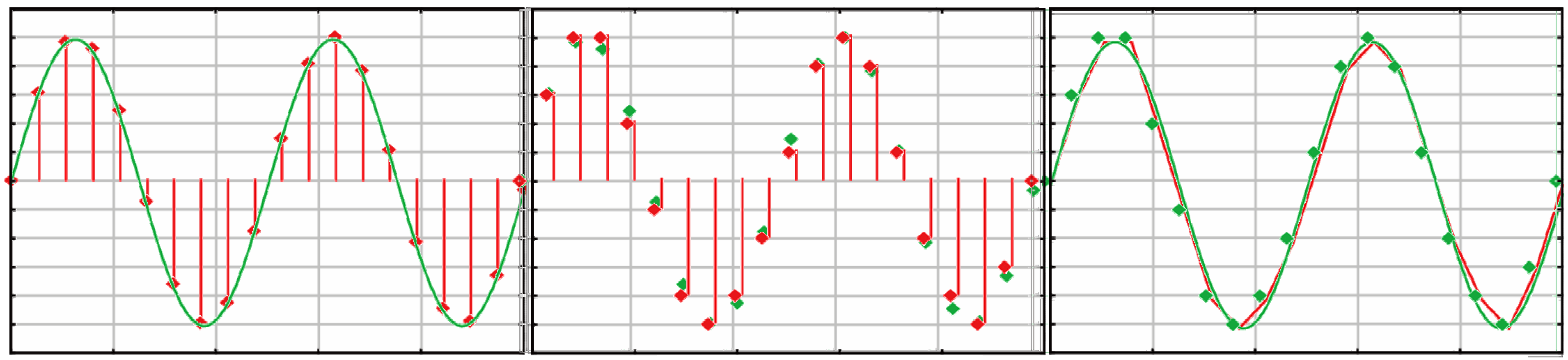
A/D D/A conversion

- **A/D D/A conversion are foundation for a variety of applications,**

  e.g., audio, image and communication etc.
- **Quantization error is inevitable.**



sampling **quantization** **D/A conversion**

Conventional methods of A/D D/A conversion

- **PCM**

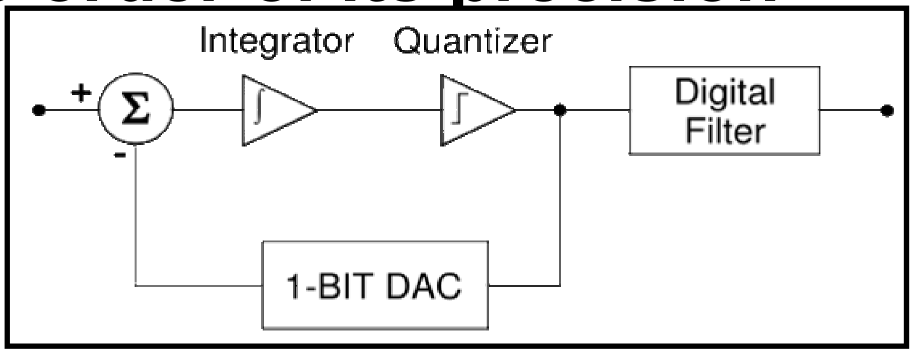
  has high precision, but doesn't have stability.

- **Σ Δ modulation**

  has stability, but the order of its precision is lower than PCM .

β -encoder

high precision, stability



The structure of Σ Δ modulator

Background

Hardware implementation

- **Inose and Yasuda '64:** $\Sigma\Delta$ modulation
- **Gray '87 :** Oversampled $\Sigma\Delta$ modulation
- **Karanicolas '93:** A 15-b 1-Msamples/s Digitally self-Calibrated Pipelaine ADC

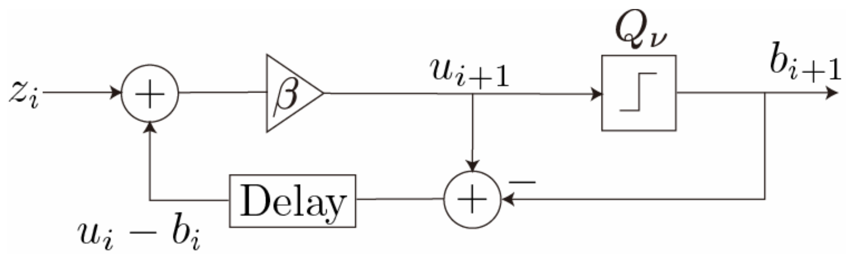
Ergodic theory

- **Renyi '57:** f-expansion
- **Parry '67:** beta-expansion
- **Erdos and Joo '90:** greedy and lazy expansion
- **Dajani '02:** (β, α) expansion

Hardware implementation
- Unsolved problem from mathematical standpoint

Ergodic theory

- Gunturk '01
- Calderbunk '02
- Daubechies '06 Ⅰ:
  anlayze stability of β -encoder,
  β of AMP and Q ν ($\cdot$), using (β , α) map
- Daubechies '06 Ⅱ: β reconstruction

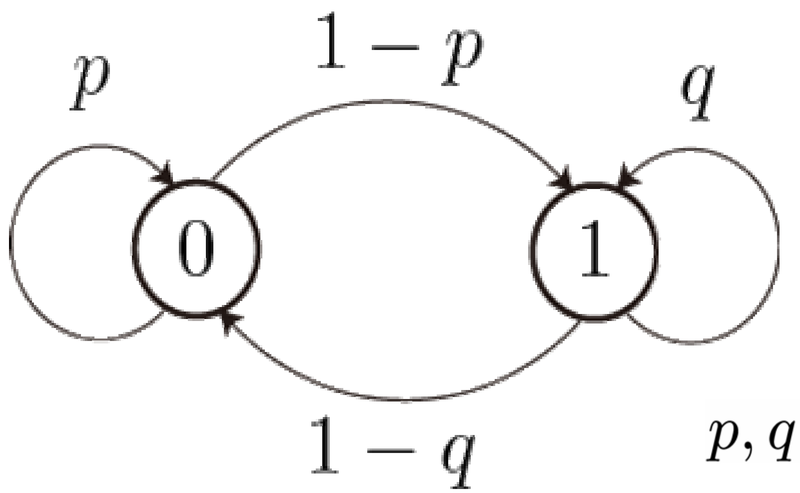


Motivation

- β -encoder generates binary sequences with **Markovity**.

  cf.) PCM generates i.i.d. binary sequences.

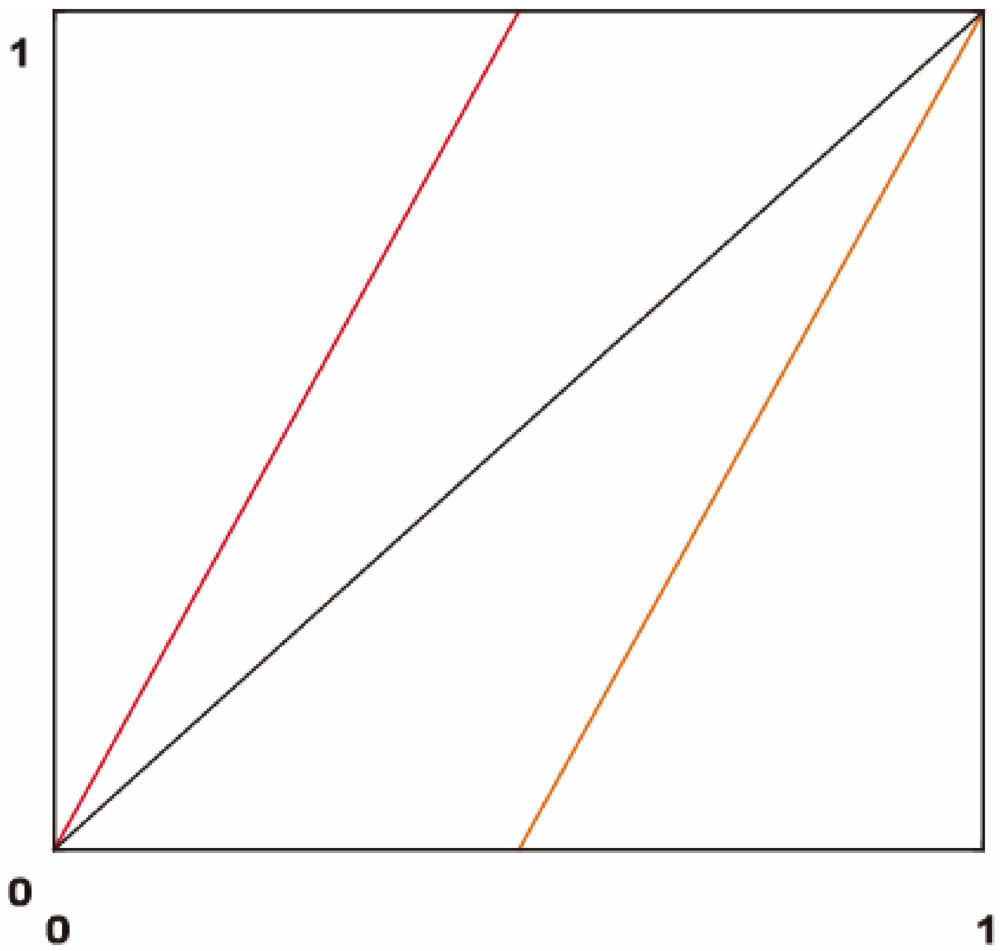
- **Does Markovity improve precision and guarantee stability? →Yes!!**



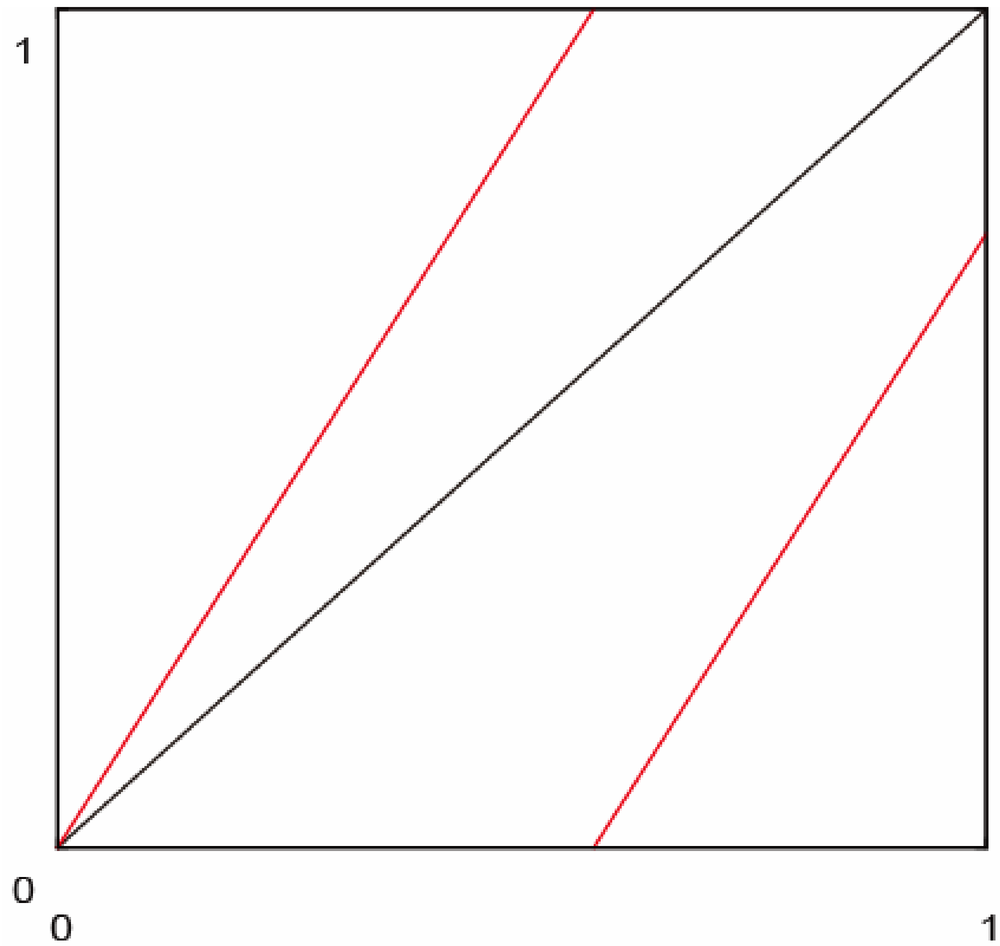
Two-state Markov chain with eigenvalue λ

$p, q \in [0, 1]$

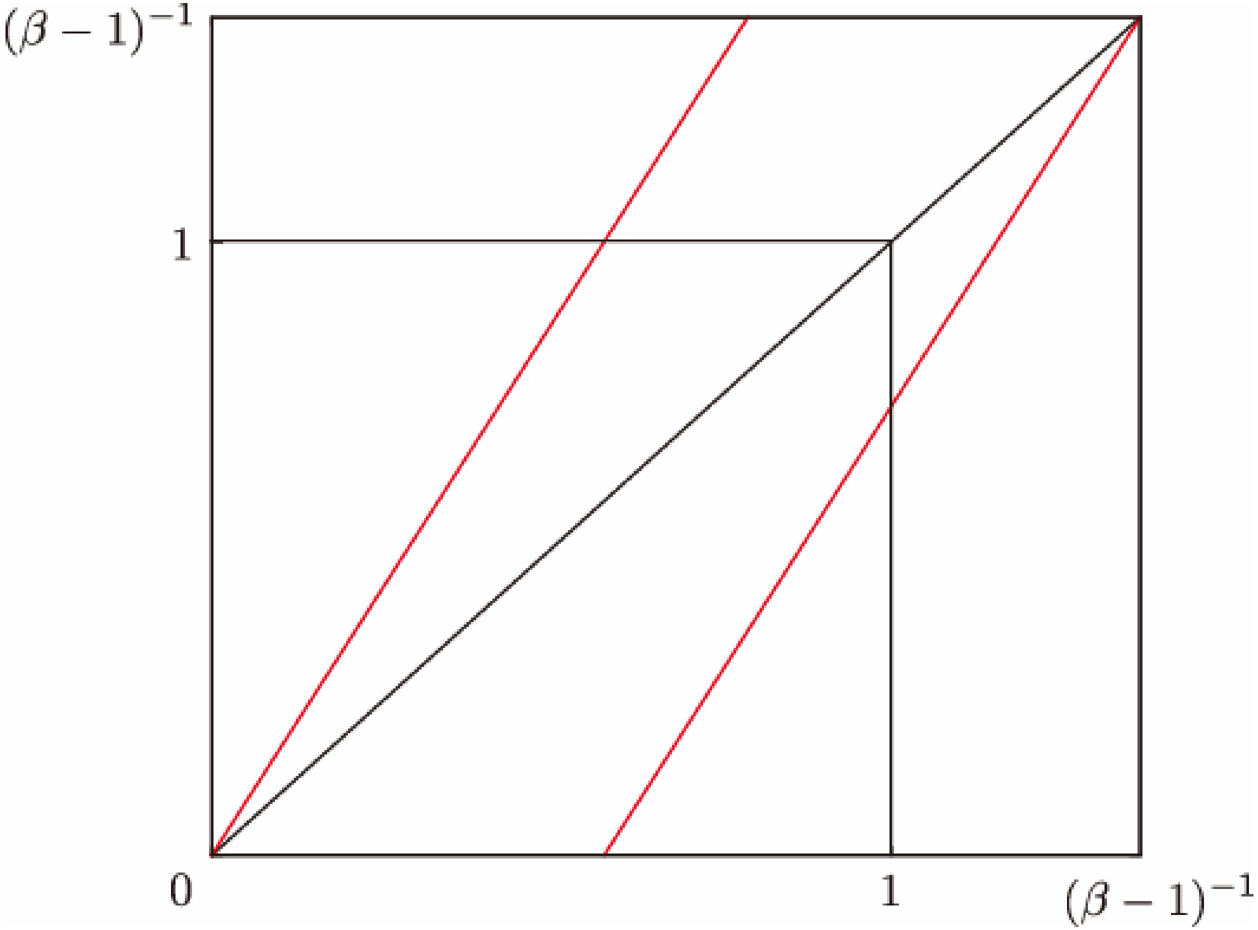
PCM (Pulse Code Modulation)



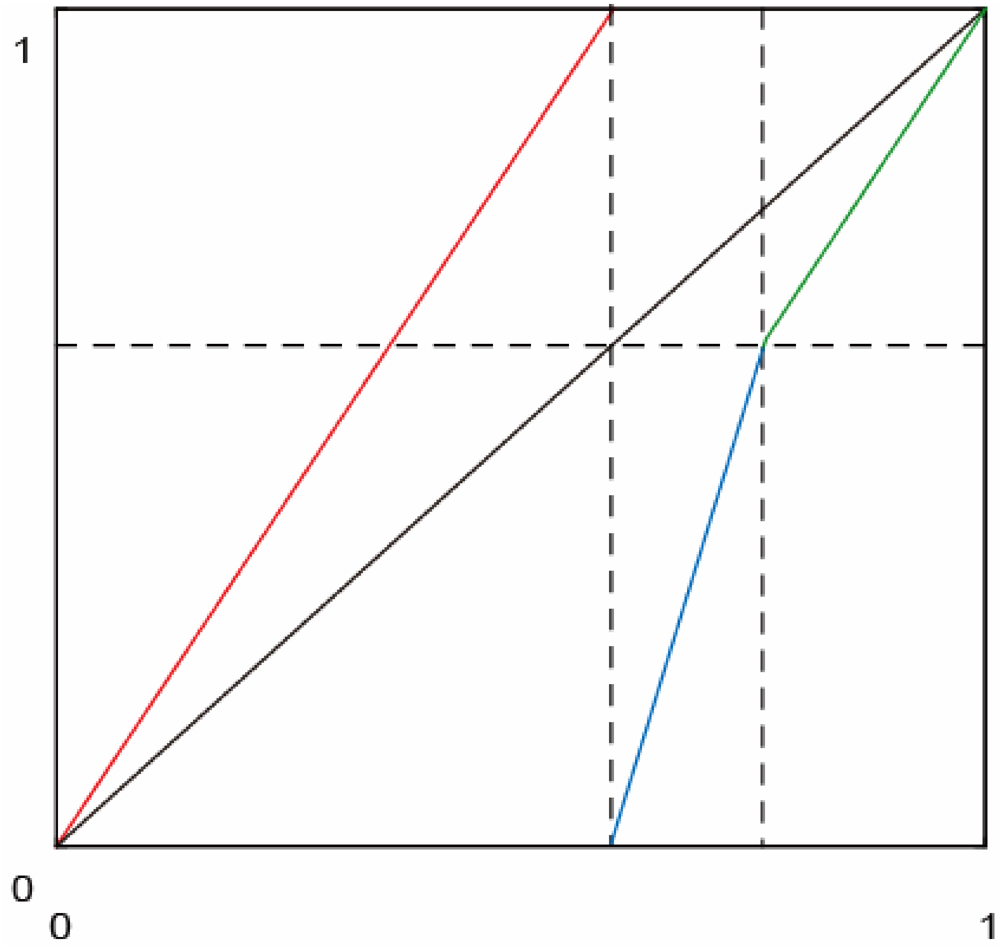
Classical β-expansion



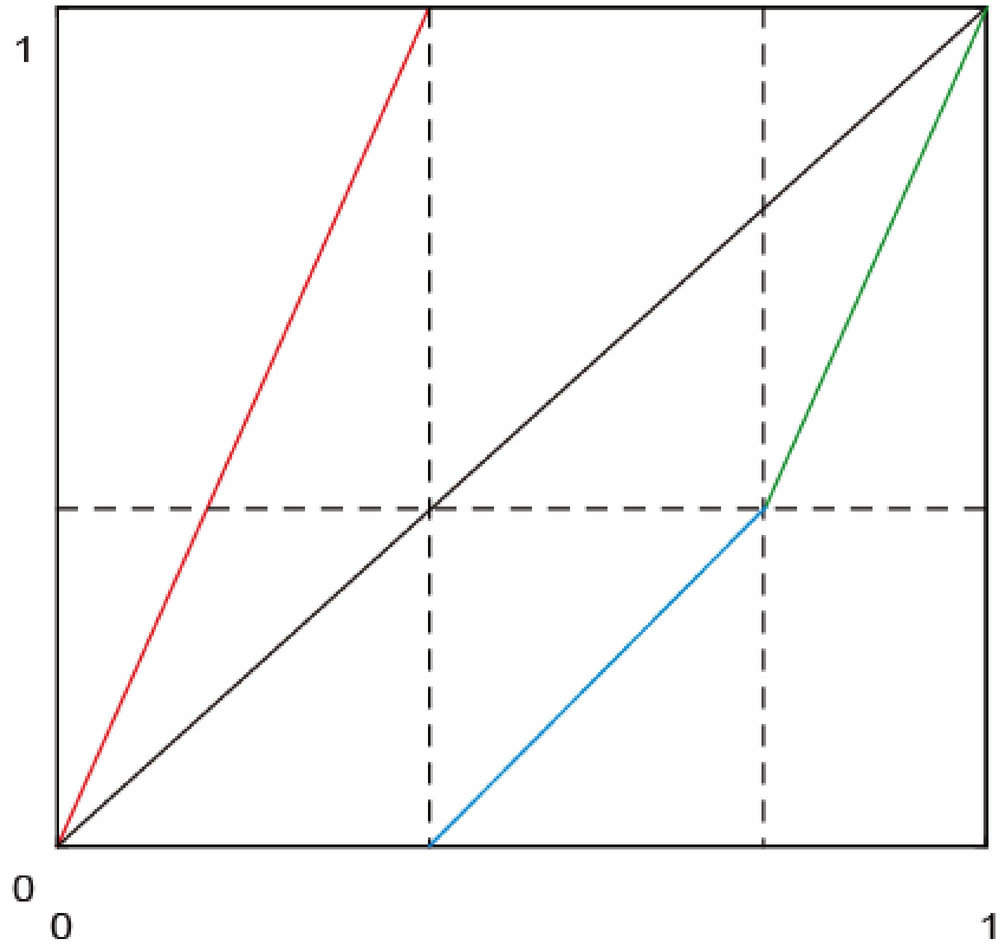
greedy and lazy expansions



マルコフマップ（λ = 0.2）



マルコフマップ（λ = −0.2)



The algorithm of beta-encoding

$$
\begin{aligned}
& u_1 = \beta y \\
& b_1 = Q_\nu(u_1) \\
for\ i \geq 1: \quad & u_{i+1} = \beta(u_i - b_i) \\
& b_{i+1} = Q_\nu(u_{i+1})
\end{aligned}
$$

quantizer

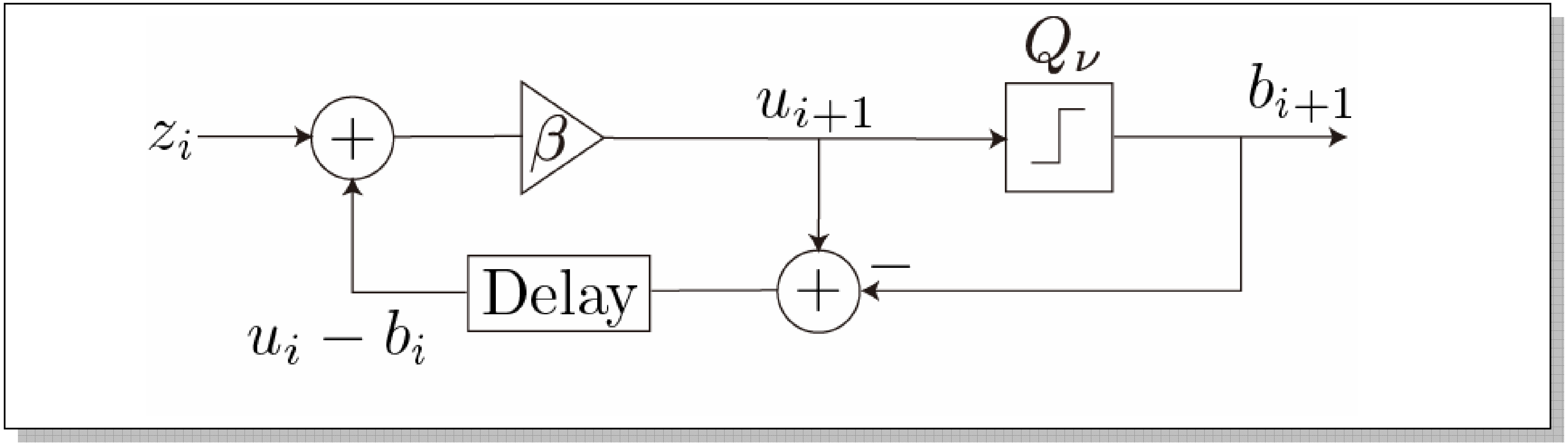
$$
Q_\nu(z) = \begin{cases} 0, & z < \nu \\ 1, & z \geq \nu \end{cases}
$$

Let $1 < \beta < 2$ and $\gamma := 1/\beta$.

Then each $y \in (0, 1)$ has a representation

$y = \sum_{i=1}^{\infty} b_i \gamma^i$ with $b_i \in \{0, 1\}$

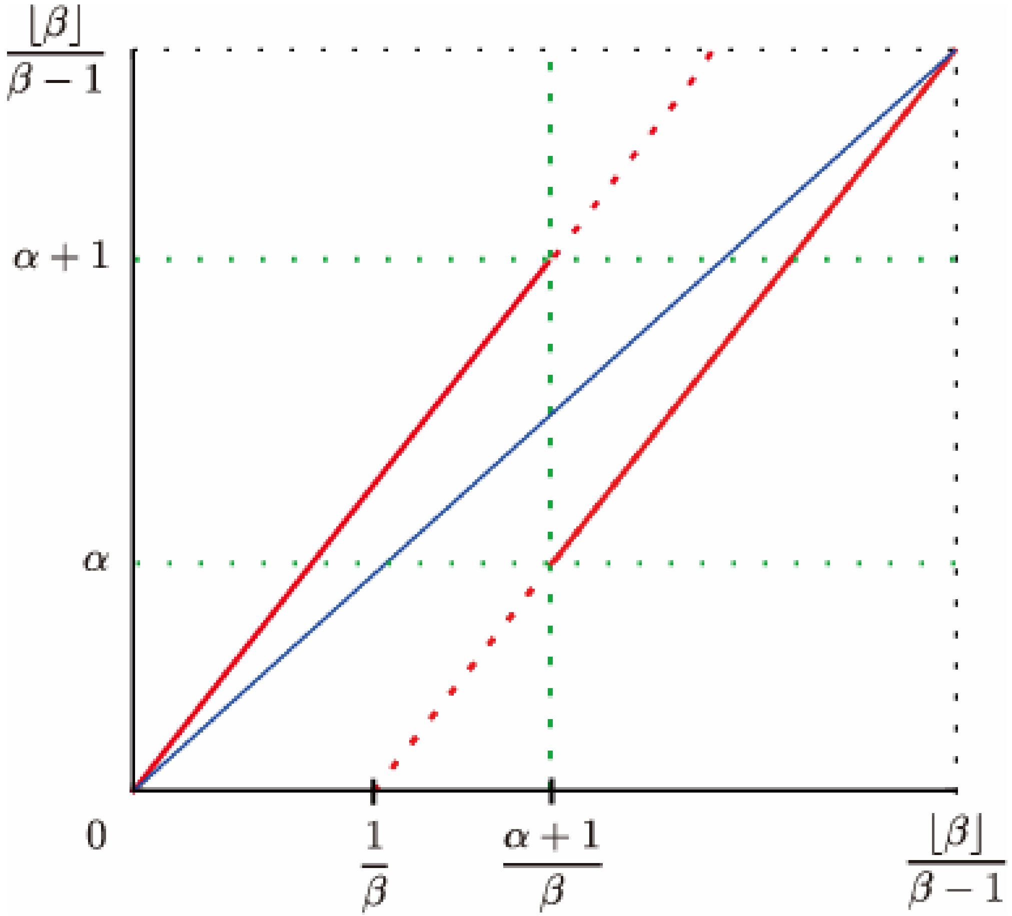
The structure of beta-encoder



$$Q_\nu(z) = \begin{cases} 0, & z < \nu \\ 1, & z \geq \nu \end{cases} \qquad \nu \in [1, (\beta - 1)^{-1}]$$

For $\nu = 1$, this is the "greedy" scheme,
for $\nu = (\beta - 1)^{-1}$, it is the "lazy" scheme,
and for $1 < \nu < (\beta - 1)^{-1}$, it is the "cautious" scheme.

The (β, α) map



For $\alpha = 0$,
"greedy".
For $\alpha = (\beta-1)^{-1} - 1$,
"lazy".
For $0 < \alpha < (\beta-1)^{-1} - 1$,
"cautious".

where $\alpha = \nu - 1$.

Main result Ⅰ: β -decoding using interval analysis

Theorem 1: The decoded value $\tilde{x}$ given by the interval analysis is defined by

$$\tilde{x} = \sum_{i=1}^{L} b_i \gamma^i + \frac{\gamma^{L+1}}{2(1-\gamma)},$$

Dust (but essential)

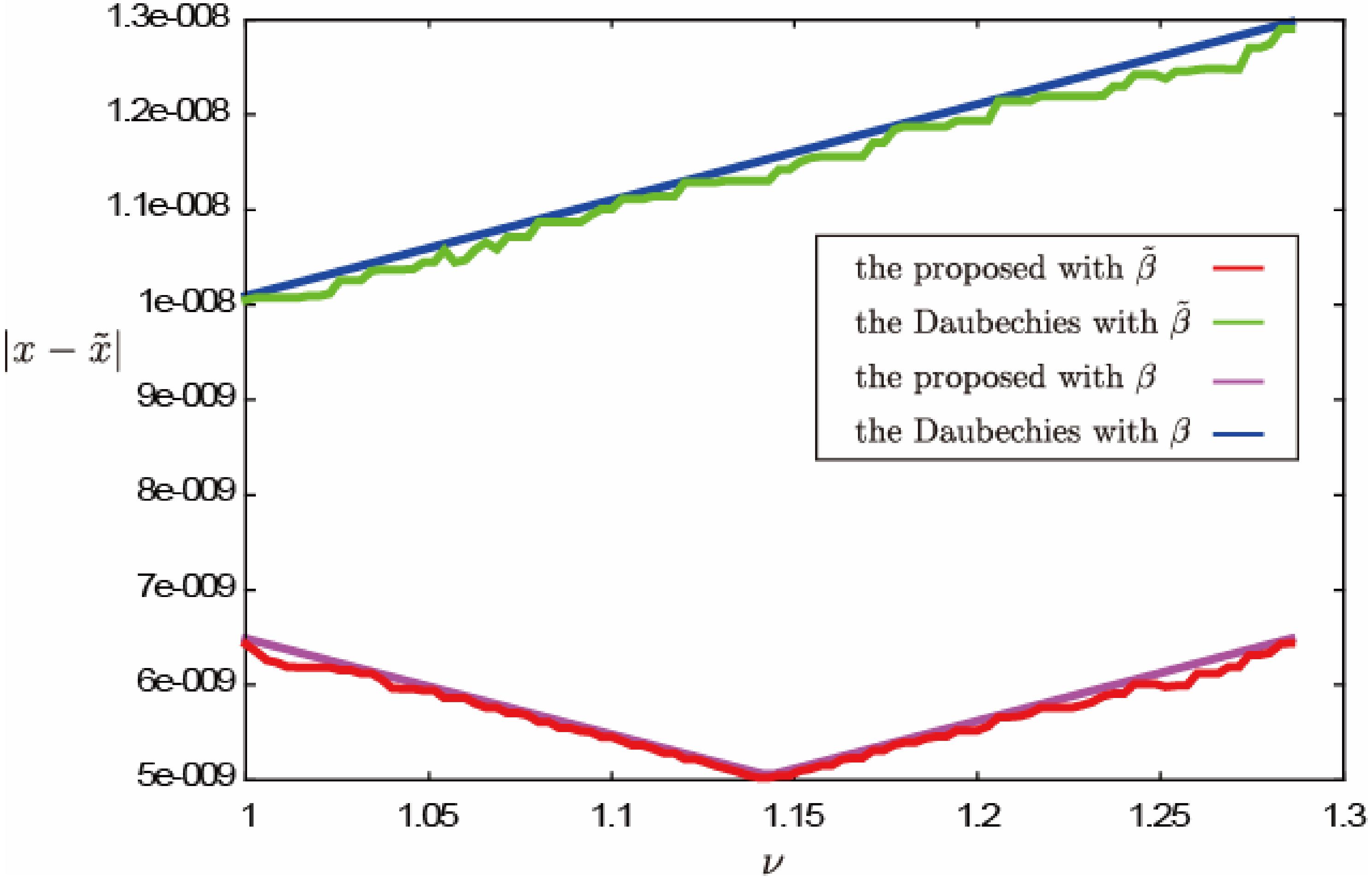
which gives

$$0 \le |x - \tilde{x}| \le \frac{(\beta-1)^{-1}\gamma^L}{2} < \gamma^L \le \underline{\nu\gamma^L} \quad \text{when } \beta > 3/2.$$

3dB improved when $\beta > 3/2$

cf.)Daubechies: $0 \le |x - \tilde{x}_{Dau}| \le \underline{\nu\gamma^L}$. $\quad \tilde{x}_{Dau} = \sum_{i=1}^{L} b_i \gamma^i$.

The precision of decoding



For $N = 32$ and $\beta = 1.77777$, the worst precision of the decoding when varing x and ν.

Proof: Define the interval

$$I_i = (\sum_{j=0}^{i} b_j\gamma^j,\ \sum_{j=0}^{i} b_j\gamma^j + \sum_{j=i+1}^{\infty} \gamma^j)$$

where $b_0 = 0$. Note that $\sum_{i=1}^{\infty} \gamma^i = (\beta - 1)^{-1}$.

Next, we use an induction argument to show that $x \in I_i$ for all i. Clearly, $x \in I_0 = (0, (\beta - 1)^{-1})$, since $x \in (0, 1)$. Suppose now that $x \in I_i$. If

$$u_{i+1} = \frac{x - \sum_{j=1}^{i} b_j\gamma^j}{\gamma^{i+1}} < \nu,$$

i.e., $b_{i+1} = 0$, then

$$x < \sum_{j=1}^{i} b_j\gamma^j + \nu\gamma^{i+1} \leq \sum_{j=0}^{i+1} b_j\gamma^j + \sum_{j=i+2}^{\infty} \gamma^j,$$

or $x \in I_{i+1}$.

If $u_{i+1} \geq \nu$, i.e., $b_{i+1} = 1, then$

$$x \geq \sum_{j=1}^{i} b_j \gamma^j + \nu \gamma^{i+1} \geq \sum_{j=0}^{i+1} b_j \gamma^j,$$

or $x \in I_{i+1}$.

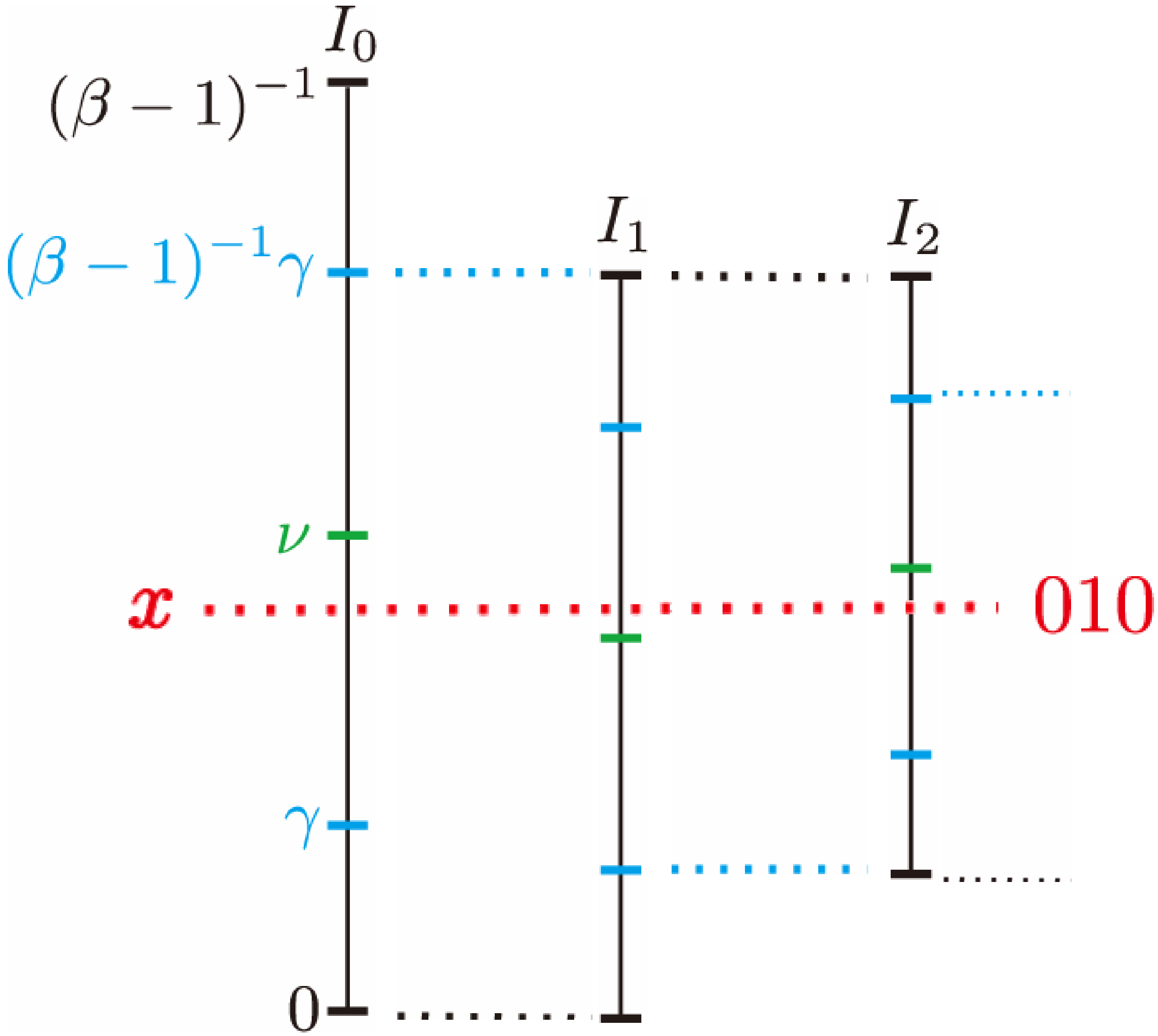
Since $x \in I_L$ and $\tilde{x} = \sum_{i=1}^{L} b_i \gamma^i + \frac{1}{2} \sum_{i=L+1}^{\infty} \gamma^i$, the approximation error is

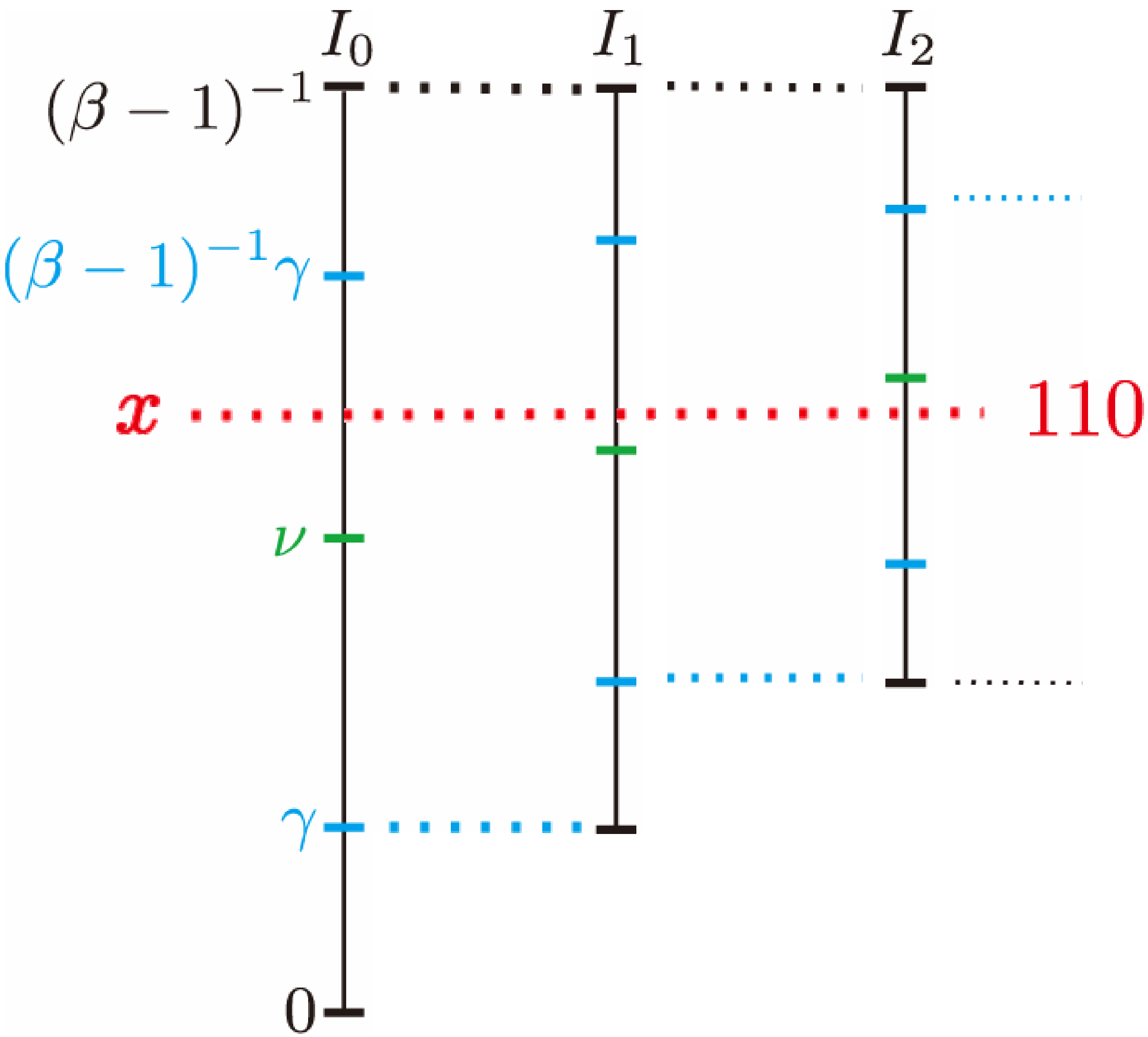
$$0 \leq |x - \tilde{x}| \leq \frac{1}{2} \sum_{i=L+1}^{\infty} \gamma^i = \frac{(\beta - 1)^{-1} \gamma^L}{2}.$$

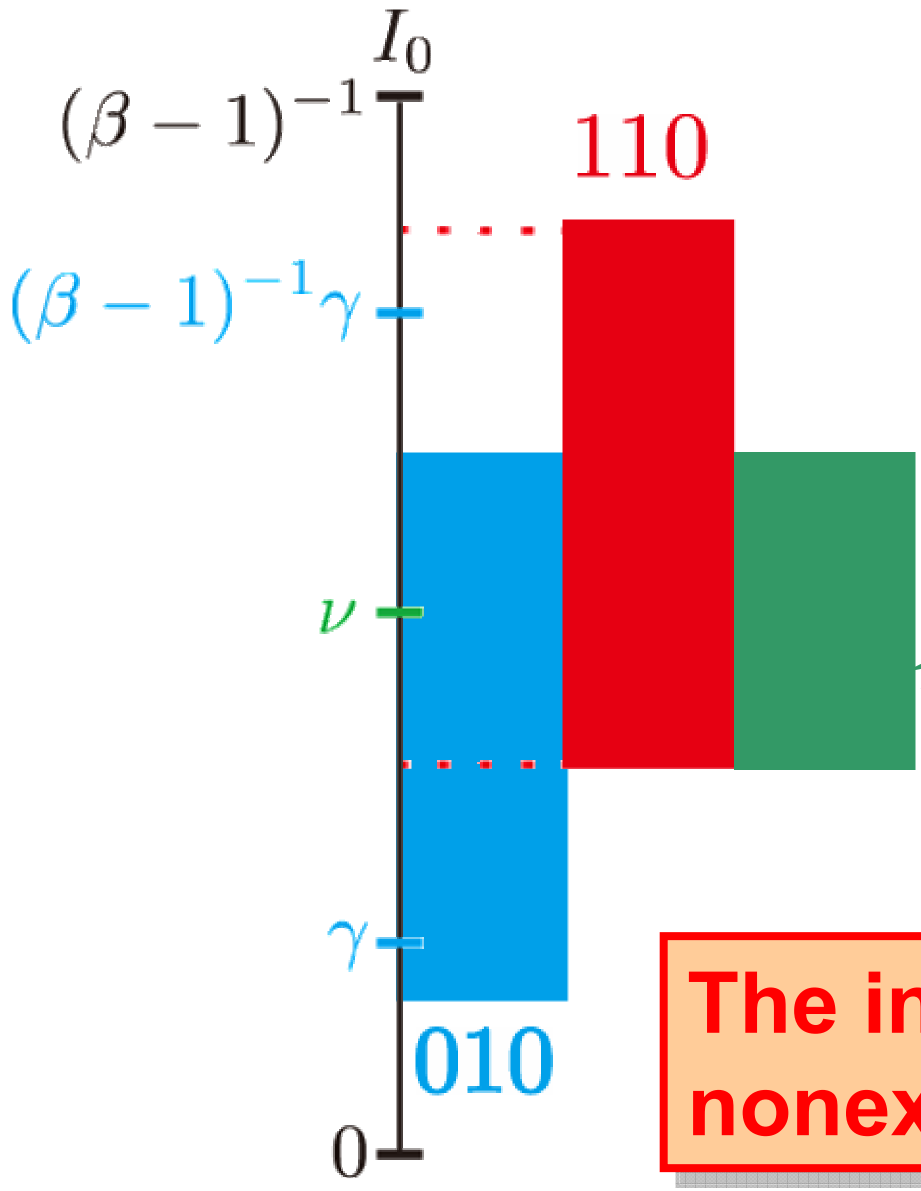
This concludes the proof. □

Dust improves the precision of decoding

The division process




I_0
I_1
I_2
$(\beta-1)^{-1}$
$(\beta-1)^{-1}\gamma$
x
110
ν
γ
0



This interval has two representation 「110」 and 「010」, i.e., redundancy.

The interval is divided into nonexclusive intervals.

Main result 2: Characteristic equation for β reconstruction

We estimate β using the sequences b_i for $x \in (0,1)$ and c_i for $y = 1 - x \in (0,1)$, $i = 1, 2, \cdots, L$, where

$$\tilde{x} = \sum_{j=1}^{i} b_j \gamma^j + \frac{\gamma^{i+1}}{2(1-\gamma)}, \quad \tilde{y} = \sum_{j=1}^{i} c_j \gamma^j + \frac{\gamma^{i+1}}{2(1-\gamma)},$$

Dust term

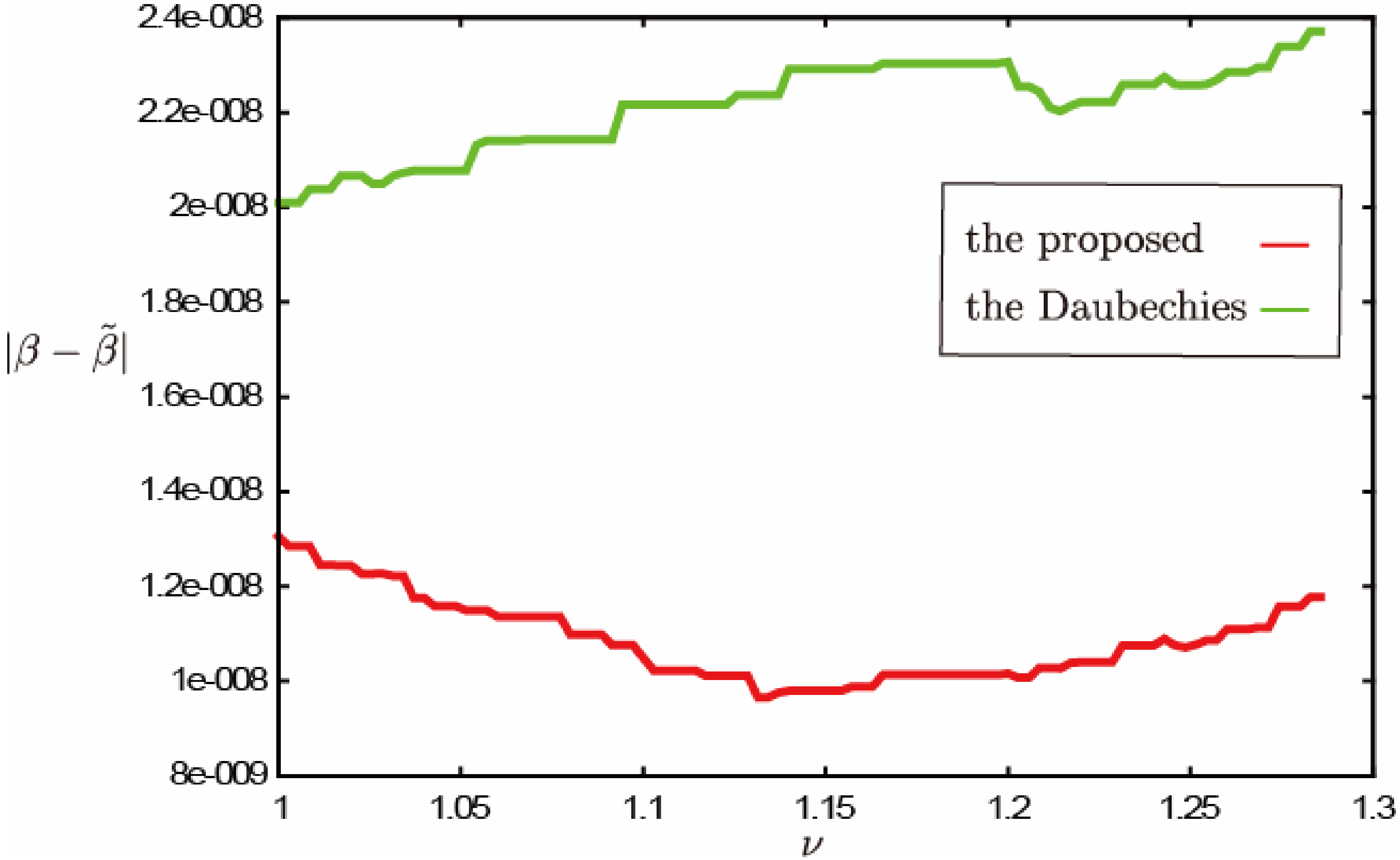
Daubechies' idea

Since $\tilde{x} + \tilde{y} = 1$, the estimated value of γ is a root of $P(\gamma)$, referred to as charasteristic equation of γ, defined by

$$P(\gamma) = 1 - \sum_{i=1}^{L} (b_i + c_i)\gamma^i - \frac{\gamma^{L+1}}{1-\gamma} = 0.$$

cf.) $P_{Dau}(\gamma) = 1 - \sum_{i=1}^{N} (b_i + c_i)\gamma^i = 0.$

The precision of beta estimation



For $N = 32$ and $\beta = 1.77777$, the worst precision of the estimation for β when varing x and ν.

Markov chain of binary sequences generated by β -encoder

PCM generates i.i.d. binary sequences,
but β -encoders does binary sequences with Markovity.

- For the invariant subinterval of β-encoder, $I = (\beta(\nu - 1), \beta\nu)$, it is very difficult to define Markov partitions of I.

⇩

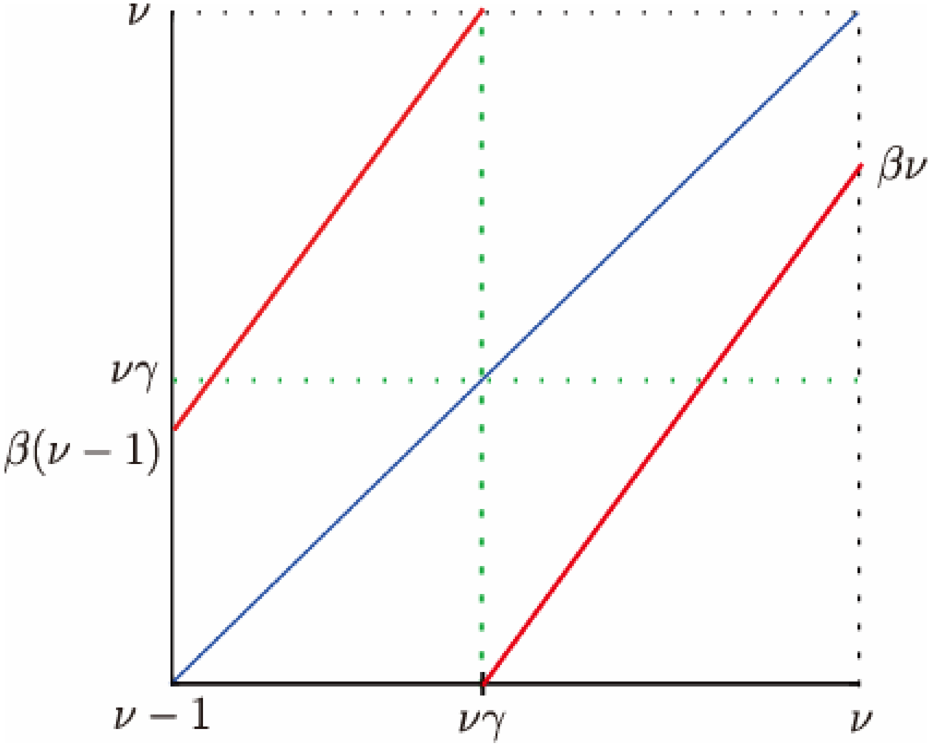
We regard b_i as output of Markov chain
and analyze eigenvalues of its Markov transition matrix.

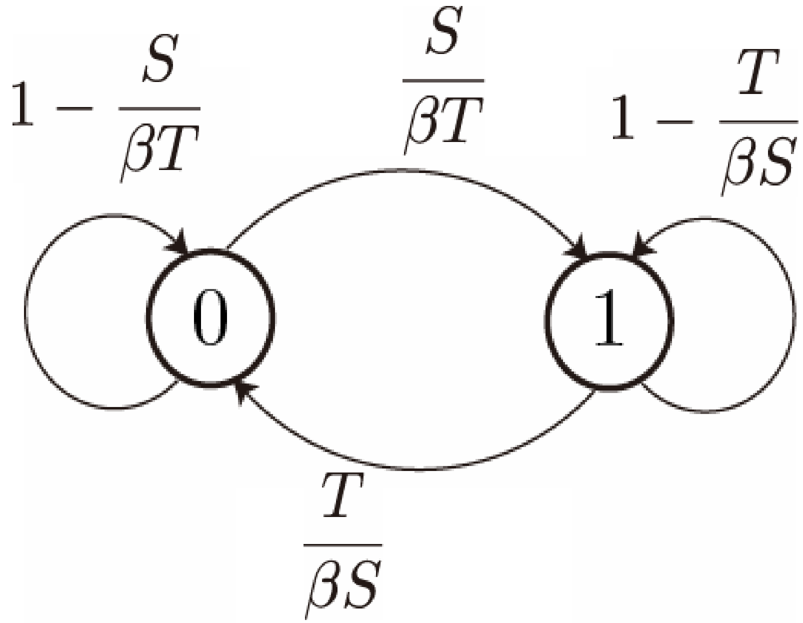
If $\frac{\beta}{\beta^2-1} \leq \nu < \frac{\beta^2}{\beta^2-1}$, the approximated transition matrix is

$$\begin{pmatrix} 1 - \frac{S}{\beta T} & \frac{S}{\beta T} \\ \frac{T}{\beta S} & 1 - \frac{T}{\beta S} \end{pmatrix}. \qquad \begin{array}{rcl} S & := & \beta\nu - \nu > 0, \\ T & := & \nu - \beta(\nu - 1) > 0. \end{array}$$

Second eigenvalue: $$\lambda = 1 - \frac{1}{\beta}(\frac{S}{T} + \frac{T}{S}) \leq 1 - \frac{2}{\beta} < 0.$$

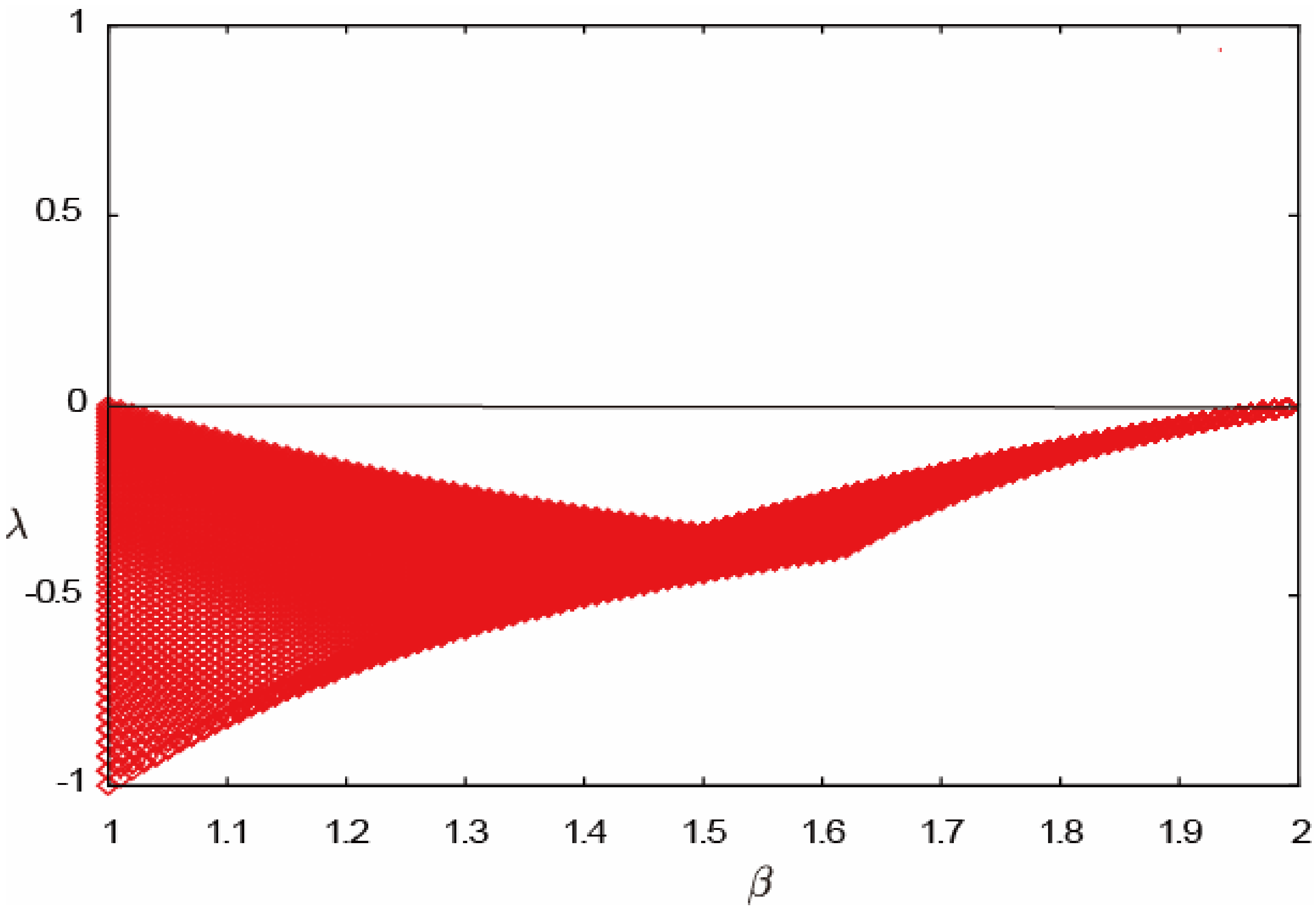
Stationary distribution: $$(\frac{T^2\beta^2}{S^2 + T^2\beta^2}, \frac{S^2}{S^2 + T^2\beta^2})$$





The invariant subinterval **The two-state Markov process**

The distribution of eigenvalues of approximated Markov transition matrix



The estimation of eigenvalue

Observing b_i for $i = 1, 2, \cdots, N$, we estimate the Markov transition matrix of beta-encoder. Let $n_{00}, n_{01}, n_{10}, n_{11}$ be defined by,

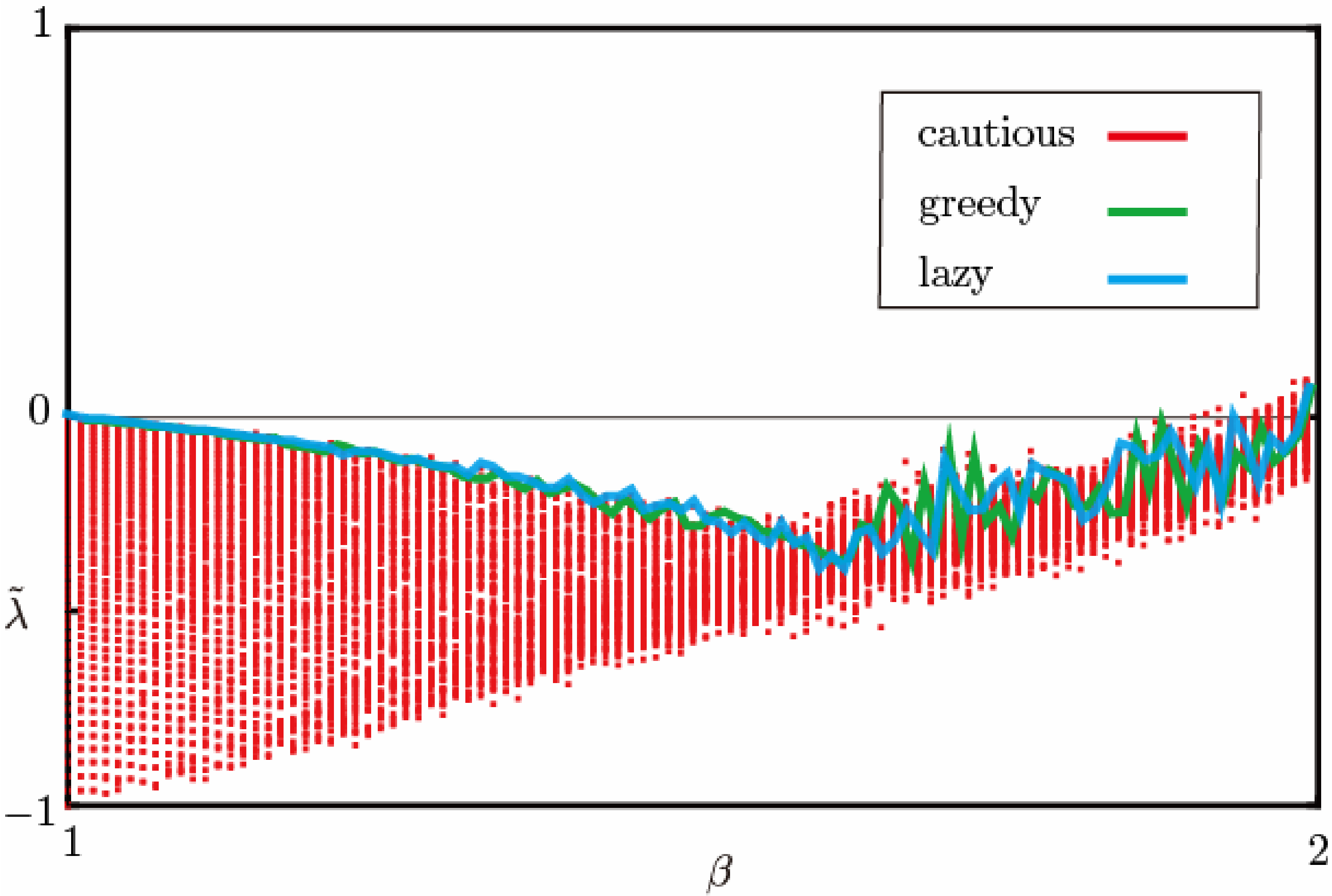
$$n_{00} := \sum_{i=1}^{N-1} (1-b_i)(1-b_{i+1}), \qquad n_{01} := \sum_{i=1}^{N-1} (1-b_i)b_{i+1},$$

$$n_{10} := \sum_{i=1}^{N-1} b_i(1-b_{i+1}), \qquad n_{11} := \sum_{i=1}^{N-1} b_i b_{i+1}.$$

The estimated Markov transition matrix is represented by

$$\begin{pmatrix} \frac{n_{00}}{n_{00}+n_{01}} & \frac{n_{01}}{n_{00}+n_{01}} \\ \frac{n_{10}}{n_{10}+n_{11}} & \frac{n_{11}}{n_{10}+n_{11}} \end{pmatrix}.$$

The distribution of eigenvalues of estimated Markov transition matrix



For $N = 32$ and $x = \pi/10$, the distribution of estimated second eigenvalues when varing β and ν.

Conclusion

Dust term $\frac{\gamma^{L+1}}{2(1-\gamma)}$ gives

- improved the precision of β -encoder up to 3dB when $\beta > 3/2$.
- derived the characteristic equation for β reconstruction.
- recommended the setting value of ν is the midpoint $\frac{(\beta-1)^{-1}+1}{2}$, neither greedy value $\nu = 1$ nor lazy value $\nu = (\beta - 1)^{-1}$.